maximum The value of $\varkappa$ is about 20, and the value of $4\pi\eta_3/\gamma$ is about 0 074

Incidentally, before going on to consider the simultaneous optimization of $\varkappa$ and σ, let us estimate the time to damp the librations with this value of $4\pi\eta_3/\gamma$ γ can be made 0 1 or larger with available materials Using $\gamma = 0\ 1$, $\eta_3 = 0\ 00059$ The amplitude thus decays by a factor of e when $\tau = 1700$ rad This corresponds to about 270 orbital periods, or about 20 days for a near-earth satellite This time is tolerable for many applications, even though it corresponds to σ different from the optimum

Now consider what happens to the curves in Fig 4 as σ is changed From Table 2, we see that η_2 is independent of σ as $\varkappa$ approaches either 3 or ∞, and hence depends but slightly upon σ for any $\varkappa$ In fact, η_2 need not concern us further Within the range of $\varkappa$ in Fig 4, $\eta_1 > \eta_3$ However, since η_1 goes as $1/\varkappa$ and η_3 as $1/\varkappa^{1/2}$ for larger $\varkappa$, from Table 2, η_1 will ultimately become less than η_3 For $\sigma = 2$, η_1 crosses η_3 to the right of the η_3 maximum.

We would like to make η_3 larger than it is for $\sigma = 2$ From the series listed in Table 2, we see that this requires making σ smaller Also, from Table 2, making σ smaller does not change η_1 for small $\varkappa$, but decreases η_1 proportionally to σ for large $\varkappa$ Hence, as we decrease σ, the point at which $\eta_1 = \eta_3$ moves to the left, approaching the maximum of η_3, at the same time that the maximum value of η_3 is increasing Hence, the optimum condition occurs§ when the curve of η_1 crosses the curve of η_3 at the maximum of η_3 This condition fixes both σ and $\varkappa$

The optimum values are approximately

$$\sigma = 1\ 3 \qquad \varkappa = 27 \qquad \eta_1 = \eta_3 = 0\ 203 \tag{20}$$

The characteristic damping time, with $\nu = 0\ 1$ as before, is $\tau = 619$ rad, corresponding to 98 orbits or about 1 week As R E Fischell of the Applied Physics Laboratory will show in a forthcoming paper, values of γ near 0 5 are probably achievable This value of γ reduces the damping time to about 20 orbits, or 1 4 days

§ This argument is not rigorous Certainly, with the entire η_3 curve rising with decreasing σ, we can improve matters by decreasing σ as long as η_1 crosses η_3 to the right of the η_3 maximum Now denote the common value of η_1 and η_3, when the intersection occurs at the η_3 maximum, by η_c If we further decrease σ, since the entire η_3 curve is still rising, the intersection of η_1 and η_3 might correspond to a larger value than η_c, even though the intersection is to the left of the maximum in η_3 when η_3 is considered as a function of $\varkappa$ alone Much laborious calculation would be required to settle this point It is clear that the criterion described in the text is close to the optimum

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JANUARY 1964 AIAA JOURNAL VOL 2, NO 1

Optimal Programming Problems with Inequality Constraints II: Solution by Steepest-Ascent

WALTER F DENHAM* AND ARTHUR E BRYSON JR †

Raytheon Company, Bedford, Mass , and Harvard University, Cambridge, Mass

A substantial class of optimal programming problems has been treated successfully by the steepest-ascent computation procedure introduced in Refs 1 and 2 Inequality constraints on functions of the control and/or state variables have been treated by several investigators through the use of integral penalty functions In this paper such constraints are included in a manner which is naturally consistent with the necessary conditions for an extremal solution Calculation of the influence functions on terminal quantities takes into account that portions of the path are on the constraint boundary An appropriately modified version of the steepest-ascent technique of Ref 2 is then constructed Numerical solutions to two atmospheric entry trajectory problems are given, using both the direct method of this paper and the penalty function method

1 Introduction

A SUBSTANTIAL class of optimal programming problems has been treated successfully by the steepest-ascent computation procedure proposed in Refs 1 and 2 These problems involve determining control variable programs to maximize a terminal quantity, with certain initial and terminal quantities specified Solutions are obtained by successive improvements in the control variable programs

This paper presents an extension of the procedure to handle problems in which there is an inequality constraint on a

Presented as Preprint 63-78 at the IAS 31st Annual Meeting, New York, January 20–23 1963; revision received October 2, 1963 This work was wholly sponsored by Raytheon Company

* Research Fellow, Harvard University, and Consultant, Raytheon Company

† Professor of Mechanical Engineering, Harvard University, and Consultant, Raytheon Company Member AIAA

function of the control and/or state variables ‡ The necessary conditions for a stationary solution in these cases, extending the work of Gamkrelidze and Berkovitz, are given in Ref 4 Previous computation schemes have used a "penalty function," requiring the introduction of an auxiliary state variable One such scheme[5] uses an auxiliary state variable which is the maximum value of the constraint function up to the present time If the terminal value of this quantity exceeds the prescribed limit, it is subsequently treated as an additional terminal constraint The other scheme[6 7] uses an auxiliary state variable which is the integral of a quadratic measure of the violation of the inequality constraint up to the present time The terminal value of this quantity, if nonzero, is treated as an additional terminal constraint and brought as close to zero as is necessary to provide a satisfactorily small violation of the inequality constraint The scheme presented here is believed to be more direct and natural than introducing an auxiliary state variable Each of the sequence of improving solutions is forced to satisfy the inequality constraints either completely or partially Typically, this results in one or more periods on the constraint boundary A modified steepest-ascent technique is used; as in Ref 2, this technique relies upon the impulse response function for computing changes in the control program from step to step in the improvement sequence The impulse response function is computed from sets of influence functions which are solutions to differential equations adjoint to the equations describing perturbations about the present solution In general, during the periods on the constraint boundary, certain terms must be added to the adjoint equations to properly account for the presence of the constraint In addition, the influence functions may be discontinuous at points where the solution goes onto the constraint boundary

This method offers three major advantages over the penalty function approach: 1) an additional state variable is not required, 2) each of the sequence of improving solutions satisfies the constraint either completely or partially, and 3) "improvements" in the control program are not required for the periods on the constraint boundary, making possible more rapid convergence toward the optimal program

Numerical solutions to two example problems are presented to illustrate the advantages just listed Both problems involve an Apollo-type vehicle entering the atmosphere at 400,000 ft alt, $-7\ 5°$ flight-path angle, and 36,000 ft-sec^{-1} velocity In the first problem the velocity is maximized at zero flight-path angle, 250,000 ft alt after an initial pull-up The inequality constraint is a limitation on the total aerodynamic deceleration to no more than 5 g's; the aerodynamic deceleration is a function of both control and state variables In the second problem the total surface range is maximized, subject to the inequality constraint that the vehicle stay below a prescribed altitude after the initial pull-up; altitude is one of the state variables For comparison, the numerical examples were also carried out using the penalty function method for handling the inequality constraints

2 Optimal Programming Problem with an Inequality Constraint

The problem to be solved numerically is the same as the one treated theoretically in Ref 4 We wish to determine $\alpha(t)$ in the interval $t_0 \leq t \leq t_f$ so as to maximize (or minimize)

$$J = \phi[\mathbf{x}(t_f),t_f] \tag{2 1}$$

subject to the constraints

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t),\alpha(t),t] \tag{2 2}$$

$$\mathbf{M} = \mathbf{M}[\mathbf{x}(t_f),t_f] = 0 \tag{2 3}$$

$$t_0 \text{ and } \mathbf{x}(t_0) \text{ given} \tag{2 4}$$

$$C(\mathbf{x},\alpha,t) \leq 0 \tag{2 5}$$

or

$$S(\mathbf{x},t) \leq 0 \tag{2 6}$$

where

t = the independent variable, hereafter called time

$\dot{(\ )} = d/dt(\ )$

$\alpha(t)$ = a scalar control variable which may be freely chosen within the limits imposed by $C \leq 0$ or $S \leq 0$§

$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is an n vector of state variable histories, which result from a choice of $\alpha(t)$ and given values of $\mathbf{x}(t_0)$

$\mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ is an n vector of known functions of $\mathbf{x}(t)$, $\alpha(t)$, and t

ϕ = the performance index and is a known function of $\mathbf{x}(t_f)$ and t_f

$\mathbf{M} = \begin{bmatrix} M_1 \\ \vdots \\ M_p \end{bmatrix}$ is a p vector of terminal constraint functions, each of which is a known function of $\mathbf{x}(t_f)$ and t_f; p must be $\leq n$

C = a scalar function of $\mathbf{x}(t)$, $\alpha(t)$, and t§

S = a scalar function of $\mathbf{x}(t)$ and t§

There is no direct method of solution known for this problem We shall present the appropriate version of the steepest-ascent computation technique for successive approximation to a solution

3 Determination of the Influence Functions

The steepest-ascent technique starts with a nonoptimal solution of the equations of motion of the system, which usually does not satisfy all the terminal constraints or the instantaneous inequality constraint, and systematically improves successive solutions until they approach the desired extremal The cornerstone of the technique is the impulse response function for changes in the control program, which is obtained directly from the influence functions In this section we present the method of computing the influence functions when there is an inequality constraint

First, we examine the $S < 0$ constraint relation We suppose that $d^qS/dt^q = S^{(q)}[\mathbf{x},\alpha,t]$ is the first derivative of S to be explicit function of α This we call a qth order state variable inequality constraint In general, when a point $S = 0$ is reached at time, say τ, it is impossible to maintain $S = 0$ for $t > \tau$ if q is greater than one This immediately follows upon recognizing that $S^{(q)}$ is the rate of change of $S^{(q-1)}$ and cannot control the rate of change of any lower derivative directly If the lowest nonzero derivative of S is positive when $S = 0$ is reached, S will violate the constraint

For the general member of the sequence of improving solutions the appropriate state variable constraint is

$$S^{(q-1)}[\mathbf{x}(t),t] \leq 0 \tag{3 1}$$

This constraint, of course, is only imposed if $S \gtrapprox 0$ The constraint boundary is maintained by choosing α to keep

$$S^{(q)}[\mathbf{x}(t),\alpha(t),t] = 0 \tag{3 2}$$

‡ Problems with more than one inequality constraint and more than one control variable are considered in Ref 3

§ See footnote on this page, left column

Thus, at some time t_1 (with $S \gtrapprox 0$#), defined by reaching

$$S^{(q-1)}[\mathbf{x}(t_1),t_1] = 0 \tag{3 3}$$

we begin using (3 2), $t > t_1$, to keep $S^{(q-1)} = 0$ until some time t_2 (which may be chosen, although not entirely freely) when $\alpha(t_2)$ causes $S^{(q)}$ to become negative Then $S^{(q-1)}$ becomes negative immediately thereafter

The first step in calculating a nominal path and its influence functions is the selection of a nominal control program $\alpha(t)$ Using this nominal program (with the given initial state) the equations of motion are integrated until (with $S \lessapprox 0$)

$$U[\mathbf{x}(t),t]_{t=t_1} = S^{(q-1)}[\mathbf{x}(t_1),t_1] = 0 \tag{3 4}$$

The control variable inequality constraint $C(\mathbf{x},\alpha,t) \leq 0$ is essentially a trivial special case and will not be explicitly treated again until Sec 5 For $t_1 < t \leq t_2$, $S^{(q)} = 0$ is used to determine $\alpha(t)$ (The question of selecting t_2 is discussed later) At t_2 the constraint boundary is left and the integration is continued with the nominal control program until the terminal point is reached One of the terminal constraint functions, say M_p, is used to determine t_f We denote it

$$\Omega[\mathbf{x}(t),t]_{t=t_f} = M_p[\mathbf{x}(t_f),t_f] = 0 \tag{3 5}$$

Thus, a nominal solution is obtained

To use steepest-ascent we need influence functions on several terminal quantities Let Z stand for a typical terminal quantity, a function of $\mathbf{x}(t)$ and t evaluated at t_f We have that the change in Z is

$$dZ = \left(\frac{\partial Z}{\partial \mathbf{x}}\right)_f d\mathbf{x}_f + \left(\frac{\partial Z}{\partial t}\right)_f dt_f \tag{3 6}$$

where $\mathbf{x}_f$ stands for $\mathbf{x}(t_f)$ and the partial derivatives are evaluated using the nominal t_f and $\mathbf{x}_f$ To predict the changes $d\mathbf{x}_f$ and dt_f we use the linearized perturbation equations

$$\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\,\delta\mathbf{x} + \frac{\partial \mathbf{f}}{\partial \alpha}\,\delta\alpha \tag{3 7}$$

where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_n} \\ & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \frac{\partial \mathbf{f}}{\partial \alpha} = \begin{bmatrix} \frac{\partial f_1}{\partial \alpha} \\ \\ \frac{\partial f_n}{\partial \alpha} \end{bmatrix}$$

are evaluated along the nominal path In the interval $t_1 < t \leq t_2$ a perturbed solution must satisfy

$$\delta S^{(q)} = \frac{\partial S^{(q)}}{\partial \mathbf{x}}\,\delta\mathbf{x} + \frac{\partial S^{(q)}}{\partial \alpha}\,\delta\alpha = 0 \tag{3 8}$$

where $\partial S^{(q)}/\partial \mathbf{x} = [\partial S^{(q)}/\partial x_1, \quad , \partial S^{(q)}/\partial x_n]$ and $\partial S^{(q)}/\partial \alpha$ are evaluated along the nominal path It follows that the differential equations governing the influence functions are

$$\dot{\boldsymbol{\lambda}} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda} = 0 \qquad S^{(q)} \neq 0 \tag{3 9}$$

$$\dot{\boldsymbol{\lambda}} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}}{\partial \alpha}\left(\frac{\partial S^{(q)}}{\partial \alpha}\right)^{-1}\frac{\partial S^{(q)}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = 0 \tag{3 10}$$

$\alpha(t)$ determined from $S^{(q)}[\mathbf{x},\alpha,t] = 0$

We wish to obtain

$$dZ = (\boldsymbol{\lambda}_{Z\Omega}^T \delta\mathbf{x})_{t=t_f} \tag{3 11}$$

and write the right-hand side of (3 11) in terms of $\delta\alpha(t)$, changes in the control program which we are free to choose The required terminal value of $\boldsymbol{\lambda}_{Z\Omega}$ is (see Ref 2, Appendix D)

$$\boldsymbol{\lambda}_{Z\Omega}^T(t_f) = \left(\frac{\partial Z}{\partial \mathbf{x}} - \frac{\dot{Z}}{\dot{\Omega}}\frac{\partial \Omega}{\partial \mathbf{x}}\right)_{t=t_f} \tag{3 12}$$

In the interval $t_2 \leq t \leq t_f$ we can use (3 7) and (3 9) to directly show that

$$(d/dt)(\boldsymbol{\lambda}^T\delta\mathbf{x}) = \boldsymbol{\lambda}^T(\partial\mathbf{f}/\partial\alpha)\delta\alpha \tag{3 13}$$

Using boundary conditions (3 12) we can integrate (3 13) from t_f backward to t_2 to obtain

$$(\boldsymbol{\lambda}_{Z\Omega}^T\delta\mathbf{x})_{t=t_f} = (\boldsymbol{\lambda}_{Z\Omega}^T\delta\mathbf{x})_{t=t_2} + \int_{t_2}^{t_f} \boldsymbol{\lambda}_{Z\Omega}^T \frac{\partial \mathbf{f}}{\partial \alpha}\,\delta\alpha\,dt \tag{3 14}$$

In the interval $t_1 < t \leq t_2$ we can use Eqs (3 7, 3 8, and 3 10) to show that

$$(d/dt)(\boldsymbol{\lambda}^T\delta\mathbf{x}) = 0 \tag{3 15}$$

$\boldsymbol{\lambda}$ is taken as continuous across t_2 and thus we immediately obtain

$$(\boldsymbol{\lambda}_{Z\Omega}^T\delta\mathbf{x})_{t=t_f} = (\boldsymbol{\lambda}_{Z\Omega}^T\delta\mathbf{x})_{t=t_1+} \int_{t_2}^{t_f} \boldsymbol{\lambda}_{Z\Omega}^T \frac{\partial \mathbf{f}}{\partial \alpha}\,\delta\alpha\,\mathrm{dt} \tag{3 16}$$

(We have used t_1+ instead of t_1 here in anticipation of a discontinuity across t_1) At this point we have expressed dZ in terms of the perturbation in the state at t_1+ and the perturbations in control $t \geq t_2$ If $\delta\alpha(t)$ were zero for $t > t_1$ the change in Z could be calculated from the changes $d\mathbf{x}_1$ and dt_1 From (3 16) we could identify

$$\begin{aligned} dZ &= (\partial Z/\partial \mathbf{x}_1)d\mathbf{x}_1 + (\partial Z/\partial t_1)dt_1 \\ &= \boldsymbol{\lambda}_{Z\Omega}^T(t_1 +)\delta\mathbf{x}(t_1+) \\ &= \boldsymbol{\lambda}_{Z\Omega}^T(t_1+)[d\mathbf{x}_1 - \dot{\mathbf{x}}(t_1+)dt_1] \end{aligned} \tag{3 17}$$

The substitution for $\delta\mathbf{x}(t_1+)$ in (3 17) is suitable because t_1 is not fixed; the differentials $d\mathbf{x}_1$ and dt_1 are clearly defined quantities, the same whether viewed from t_1- or from t_1+ (3 17) provides the identification

$$\left.\begin{aligned} \frac{\partial Z}{\partial \mathbf{x}_1} &= \boldsymbol{\lambda}_{Z\Omega}^T(t_1+) \\ \frac{\partial Z}{\partial t_1} &= -\boldsymbol{\lambda}_{Z\Omega}^T(t_1+)\dot{\mathbf{x}}(t_1+) \end{aligned}\right\} \tag{3 18}$$

We wish to choose $\boldsymbol{\lambda}^T(t_1 -)$ so that the change in Z due to changes $d\mathbf{x}_1$, dt_1 will be expressed in terms of $d\mathbf{x}_0$, dt_0, and $\delta\alpha(t)$, $t_0 < t < t_1$ To do this we use the form of (3 12), where $\partial Z/\partial \mathbf{x}$ and $\dot{Z}$ are evaluated at t_1-, and $U[\mathbf{x}(t_1),t_1] = 0$ replaces $\Omega[\mathbf{x}(t_f),t_f]$ as the stopping condition [see (3 4)] We have (again referring to Appendix D of Ref 2) that

$$\frac{\partial Z}{\partial \mathbf{x}_1}\,d\mathbf{x}_1 + \frac{\partial Z}{\partial t_1}\,dt_1 = \boldsymbol{\lambda}_{ZU}^T(t_1 -)[d\mathbf{x}_1 - \dot{\mathbf{x}}(t_1-)dt_1] \tag{3 19}$$

if

$$\boldsymbol{\lambda}_{ZU}^T(t_1 -) = \left[\frac{\partial Z}{\partial \mathbf{x}} - \frac{\dot{Z}}{\dot{U}}\frac{\partial U}{\partial \mathbf{x}}\right]_{t=t_1-} \tag{3 20}$$

The partial derivatives of Z at t_1 are evaluated by (3 18) Using them we obtain

$$\begin{aligned} \dot{Z}(t_1-) &= \frac{\partial Z}{\partial \mathbf{x}_1}\,\dot{\mathbf{x}}(t_1-) + \frac{\partial Z}{\partial t_1} \\ &= \boldsymbol{\lambda}_{Z\Omega}^T(t_1+)\dot{\mathbf{x}}(t_1-) - \boldsymbol{\lambda}_{Z\Omega}^T(t_1+)\dot{\mathbf{x}}(t_1+) \end{aligned} \tag{3 21}$$

Since $U = S^{(q-1)}$, we have $\dot{U} = S^{(q)}$ Substituting these,

Usually we would not define the time t_1 unless S were at least close to zero, but S may be slightly less than zero at $t = t_1$ on any given nominal path, since it is being forced to zero

(3 18) and (3 21) into (3 20), we obtain

$$\lambda_{ZU}^T(t_1-) = \lambda_{Z\Omega}^T(t_1+) \times \left[\mathbf{I} - \frac{\dot{\mathbf{x}}(t_1-) - \dot{\mathbf{x}}(t_1+)}{S^{(q)}(t_1-)}\left(\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right)_{t=t_1}\right] \quad (3\ 22)$$

Examination of (3 22) shows that unless $S^{(q)}$ goes to zero with $S^{(q-1)}$ at $t = t_1-$, the discontinuity in $S^{(q)}$ necessarily implies a discontinuity in $\dot{\mathbf{x}}$ and thus a jump in λ Even if $S^{(q)}(t_1-)$ is zero we may expect a jump in λ In this case $\dot{\mathbf{x}}(t_1+) = \dot{\mathbf{x}}(t_1-)$ and the fraction in (3 22) is indeterminate as it stands L'Hospital's rule is then used to establish the limit as $\alpha(t_1+) \to \alpha(t_1-)$

Incidentally, the jump in the Hamiltonian obeys

$$(\lambda_{ZU}^T\dot{\mathbf{x}})_{t=t_1-} = (\lambda_{Z\Omega}^T\dot{\mathbf{x}})_{t=t_1+} - \rho\left[\frac{\partial S^{(q-1)}}{\partial t}\right]_{t=t_1} \quad (3\ 23)$$

where

$$\rho = -\frac{\lambda_{Z\Omega}^T(t_1+)[\dot{\mathbf{x}}(t_1-) - \dot{\mathbf{x}}(t_1+)]}{S^{(q)}(t_1-)}$$

This relation is a consequence of (3 22), not an input to it

Using (3 22) we integrate (3 9) from t_1- back to t_0 to obtain

$$(\lambda_{Z\Omega}^T\delta\mathbf{x})_{t=t_1+} = (\lambda_{ZU}^T\delta\mathbf{x})_{t=t_1-} = (\lambda_{ZU}^T\delta\mathbf{x})_{t=t_0} + \int_{t_0}^{t_1-} \lambda_{ZU}^T \frac{\partial \mathbf{f}}{\partial \alpha}\delta\alpha\, dt \quad (3\ 24)$$

Finally, then, we combine (3 1, 3 16, and 3 24) to obtain

$$dZ = (\lambda_{ZU}^T\delta\mathbf{x})_{t=t_0} + \int_{t_0}^{t_1-} \lambda_{ZU}^T \frac{\partial \mathbf{f}}{\partial \alpha}\delta\alpha\, dt + \int_{t_2}^{t_f} \lambda_{Z\Omega}^T \frac{\partial \mathbf{f}}{\partial \alpha}\delta\alpha\, dt \quad (3\ 25)$$

With this basic relation we can now proceed to development of the steepest-ascent procedure

4 Steepest-Ascent Successive Approximation with a State Variable Inequality Constraint

The nominal $\alpha(t)$ program produces a solution which reaches $S^{(q-1)} = 0$ (with $S \gtrsim 0$) at $t = t_1$ At this time we may not expect to obtain

$$S^{(k)}[\mathbf{x}(t_1),t_1] = 0 \qquad k < q - 1 \quad (4\ 1)$$

At the terminal point $t = t_f$ we also may not expect the first $p - 1$ members of $\mathbf{M}$ to be zero We define the vector ψ, consistent with the notation of Ref 2, to be

$$\psi = \begin{bmatrix} S[x(t_1),t_1] \\ S^{(1)}[x(t_1),t_1] \\ \vdots \\ S^{(q-2)}[x(t_1),t_1] \\ M_1[x(t_f),t_f] \\ \vdots \\ M_{p-1}[x(t_f),t_f] \end{bmatrix} \quad (4\ 2)$$

(For $q = 1$, ψ contains only components of $\mathbf{M}$)

Each component of ψ is to be driven to zero in the sequence of successive improvements, while the performance index ϕ is maximized Note that $S^{(q-1)}[\mathbf{x}(t_1),t_1]$ and $M_p[\mathbf{x}(t_f),t_f]$ are automatically zero on every iteration as they are used to define t_1 and t_f respectively

The influence functions on ϕ and M_i, $i = 1,2, \ldots, p-1$, may be obtained as $\lambda_{\phi\Omega}^T$, $\lambda_{M_i\Omega}^T$ for $t > t_1$, $\lambda^T{}_U$, $\lambda_{M_iU}^T$ for $t < t_1$ by substituting ϕ and M_i for Z in the equations of the preceding section To obtain the influence functions on the $S^{(k)}[\mathbf{x}(t_1),t_1]$ we define

$$\lambda_{S^{(k)}U}^T(t_1-) = \left[\frac{\partial S^{(k)}}{\partial \mathbf{x}} - \frac{S^{(k+1)}}{\dot{U}}\frac{\partial U}{\partial \mathbf{x}}\right]_{t=t_1-} \quad (4\ 3)$$

where $U = S^{(q-1)}$ and $\dot{U} = S^{(q)}$ Equations (3 9) are integrated from t_1- back to t_0 using boundary conditions (4 3) to obtain $\lambda_{S^{(k)}U}^T$ and

$$dS^{(k)}[\mathbf{x}(t_1),t_1] = (\lambda_{S^{(k)}U}^T\delta\mathbf{x})_{t=t_0} + \int_{t_0}^{t_1-} \lambda_{S^{(k)}U}^T \frac{\partial \mathbf{f}}{\partial \alpha}\delta\alpha\, dt \quad (4\ 4)$$

We note that the influence functions on $S^{(k)}[\mathbf{x}(t_1),t_1]$ are identically zero, $t > t_1$ This is obvious in that $S^{(k)}$ at t_1 cannot be changed by anything that happens after t_1

Having obtained all the influence functions we write them together as (consistent with the notation of Ref 2)

$$\lambda_{\psi\Omega} = \begin{cases}[\lambda_{SU}, \lambda_{S^{(1)}U}, \ \ldots, \lambda_{M_1U}, \ \ldots, \lambda_{M_{p-1}U}], & t < t_1 \\ [0, \quad 0, \quad \ldots, \lambda_{M_1\Omega}, \ \ldots, \lambda_{M_{p-1}\Omega}], & t > t_1\end{cases} \quad (4\ 5)$$

with the impulse response functions $\lambda_{\psi\Omega}^T(\partial\mathbf{f}/\partial\alpha)$, where $\mathbf{\Omega}$ as a subscript represents choices of both the entry corner stopping condition U and the terminal stopping condition Ω The steepest-ascent procedure determines $\delta\alpha(t)$ to maximize (or minimize) $d\phi$ for selected values of $d\psi$, and $\delta\mathbf{x}(t_0)$ if a change in the initial conditions is desired, for a given value of

$$(dP)^2 = \int_{t_0}^{t_1} [\delta\alpha(t)]^2W(t)dt + \int_{t_2}^{t_f} [\delta\alpha(t)]^2W(t)dt \quad (4\ 6)$$

where $W(t)$ is an arbitrary positive weighting function, chosen to improve convergence Now dP must be chosen small enough that the linearization of the perturbation equations allows reasonable accuracy in the prediction of changes $d\phi$ and $d\psi$ The values of $d\psi$ and $\delta\mathbf{x}(t_0)$ which may be chosen are automatically limited by the choice of dP The change $\delta\alpha(t)$ in the control program as derived in Appendix B of Ref 2 is completely applicable here, with the modifications to be noted:

$$\delta\alpha(t) = \pm W^{-1}(t)\left[\frac{\partial \mathbf{f}}{\partial \alpha}(t)\right]^T(\lambda_{\phi\Omega}(t) - \lambda_{\psi\Omega}(t)\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi}) \times \left[\frac{(dP)^2 - d\beta^T\mathbf{I}_{\psi\psi}^{-1}\,d\beta}{I_{\phi\phi} - \mathbf{I}_{\psi\phi}^T\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi}}\right]^{1/2} + W^{-1}(t)\left[\frac{\partial \mathbf{f}}{\partial \alpha}(t)\right]^T\lambda_{\psi\Omega}(t)\mathbf{I}_{\psi\psi}^{-1}\,d\beta \qquad \begin{cases}t \le t_1 \\ t \ge t_2\end{cases} \quad (4\ 7)$$

where

$$d\beta = d\psi - \lambda_{\psi\Omega}^T(t_0)\delta\mathbf{x}(t_0)$$

$$\mathbf{I}_{\psi\psi} = \int_{t_0}^{t_1} + \int_{t_2}^{t_f} \lambda_{\psi\Omega}^T \frac{\partial \mathbf{f}}{\partial \alpha} W^{-1}\left(\frac{\partial \mathbf{f}}{\partial \alpha}\right)^T \lambda_{\psi\Omega}\, dt$$

$$\mathbf{I}_{\psi\phi} = \int_{t_0}^{t_1} + \int_{t_2}^{t_f} \lambda_{\psi\Omega}^T \frac{\partial \mathbf{f}}{\partial \alpha} W^{-1}\left(\frac{\partial \mathbf{f}}{\partial \alpha}\right)^T \lambda_{\phi\Omega}\, dt$$

$$I_{\phi\phi} = \int_{t_0}^{t_1} + \int_{t_2}^{t_f} \lambda_{\phi\Omega}^T \frac{\partial \mathbf{f}}{\partial \alpha} W^{-1}\left(\frac{\partial \mathbf{f}}{\partial \alpha}\right)^T \lambda_{\phi\Omega}\, dt$$

Note that $\lambda_{\psi\Omega}$ is an $n \times (p + q - 2)$ matrix, each column corresponding to a component of ψ

The difference here from Ref 2 is that no $\delta\alpha(t)$ is computed for $t_1 < t < t_2$ because $\delta\alpha$ is not free in that interval but is determined entirely by $\delta\mathbf{x}(t)$ which in turn results from $\delta\alpha(t)$, $t \le t_1$ Corresponding to this lack of freedom is an omission of the $t_1 < t < t_1$ interval from the integrals of the impulse response functions These integrals determine the gradients of ϕ and ψ with respect to voluntary $\delta\alpha$ and thus the impulse response functions should be integrated only when $\delta\alpha(t)$ is free

The plus sign in (4 7) is used if ϕ is to be maximized, the minus sign if ϕ is to be minimized Notice that the numerator under the square root can become negative if $d\boldsymbol{\beta}$ is too large This represents the limitation on the $d\boldsymbol{\beta}$ which can be achieved for a given step size dP For the problem of fixed initial conditions, $d\boldsymbol{\beta} = d\boldsymbol{\psi}$ The more general expression also determines the change in $\alpha(t)$ to keep $\boldsymbol{\psi} = 0$ if small $\delta\mathbf{x}(t_0)$ are imposed The other remarks of Sec 6 in Ref 2 apply unchanged to this discussion

From (4 7), $\delta\alpha(t)$ is determined except when $S^{(q)}$ equals zero, where it is determined by $\delta S^{(q)}$ equal zero From t_0 until reaching $U = 0$ the new $\alpha(t)$ program is used If t_1 occurs sooner than on the nominal, there is clearly no difficulty If U is less than zero when the nominal t_1 is reached, α may be held constant until $U = 0$ Alternative choices could be made; any one will do provided $\alpha(t)$ is continuous up until $U = 0$ On the constraint boundary the new $\alpha(t)$ is determined by $S^{(q)}$ equal to zero

At the exit corner there will be a $\delta\alpha(t_2)$ This value will make $S^{(q)}[\mathbf{x}(t_2),\alpha(t_2),t_2]$ either greater or less than zero If the calculated $\delta\alpha(t_2)$ would cause violation of the constraint, $\delta\alpha(t)$ must be chosen to keep $S^{(q)}$ equal to zero until the new $\alpha(t)$ allows $S^{(q)} \leq 0$ This increases t_2 to the time at which the calculated new $\alpha(t)$ becomes just correct for making $S^{(q)}$ equal zero

If $\delta\alpha(t_2)$ is such as to make $S^{(q)}$ less than zero at the nominal t_2, the new t_2 should be picked earlier A reasonable plan is to move t_2 back a prespecified amount τ, choosing α at the new t_2 to make $S^{(q)}$ equal to zero and letting $\alpha(t)$ be linear from the new to the old values of t_2 This plan pushes t_2 in the proper direction and keeps $\alpha(t)$ continuous across t_2, as it must be on an extremal **

The computing procedure for the optimal programming problems discussed here is the same as in Sec 7 of Ref 2, with the entry and exit corners handled as in the foregoing paragraphs

5 Numerical Treatment with a Control Variable Inequality Constraint

The control variable inequality constraint requires only very minor modifications The nominal $\alpha(t)$ program is used except where it would make $C(\mathbf{x},\alpha,t) > 0$ Let us assume that α must be determined in $t_1 \leq t \leq t_2$ so as to make $C = 0$ in that interval The influence functions satisfy

$$\dot{\boldsymbol{\lambda}} + \left(\frac{\partial \mathbf{f}^T}{\partial \mathbf{x}}\right)\boldsymbol{\lambda} = 0 \qquad C < 0 \quad (5\ 1)$$

$$\dot{\boldsymbol{\lambda}} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}}{\partial \alpha}\left(\frac{\partial C}{\partial \alpha}\right)^{-1}\frac{\partial C}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = 0 \qquad C = 0 \quad (5\ 2)$$

They are continuous across both t_1 and t_2 The rest of the analysis of Ref 2 holds, except for the omission of the $t_1 \leq t \leq t_2$ interval in the integrals of the impulse response functions as discussed in Sec 4 The technique for handling changes in the corner times is the same as given in Sec 4, for problems which are nonlinear in the control

For problems which are linear in the control variable it will usually be possible and sometimes even required that the solution be "bang-bang" Such problems are only meaningful if some constraint is placed on the magnitude of the control variable This really involves two constraints, one an upper limit, one a lower limit Consider the constraints,

$$\alpha \leq \alpha_u \qquad \alpha \geq \alpha_l \qquad (5\ 3)$$

If the nominal control program is only partly on the constraint boundaries we would expect to employ steepest-ascent as described in Sec 4 †† In some cases, however, we may anticipate (or even prove) that the optimal solution is bang-bang In such cases it would seem most appropriate to pick a bang-bang nominal

To see how to make changes in the switching times if a bang-bang nominal is used let us write the expression for dZ without invoking the inequality constraints We obtain, by integrating (3 13) with boundary conditions (3 12),

$$dZ = (\boldsymbol{\lambda}_{Z\Omega}^T\delta\mathbf{x})_{t=t_0} + \int_{t_0}^{t_f} \boldsymbol{\lambda}_{Z\Omega}^T \frac{\partial \mathbf{f}}{\partial \alpha}\,\delta\alpha\,dt \qquad (5\ 4)$$

Suppose that at switching time t_1, α goes from α_u to α_l We wish to consider a small change dt_1 in this switching time Noting that $\boldsymbol{\lambda}^T(\partial\mathbf{f}/\partial\alpha)$ is not a function of α in this discussion we can consider it constant to first order in the interval dt_1 We have that

$$\delta\alpha = (\alpha_u - \alpha_l)\ \mathrm{sgn} dt_1 \qquad (5\ 5)$$

Then the change in Z due to a change dt_1 is given by

$$dZ - (\boldsymbol{\lambda}_{Z\Omega}^T\delta\mathbf{x})_{t=t_0} = \left(\boldsymbol{\lambda}_{Z\Omega}^T\frac{\partial \mathbf{f}}{\partial \alpha}\right)_{t=t_1}(\alpha_u - \alpha_l)dt_1 \quad (5\ 6)$$

At a switching time t_j when α goes from α_l to α_u, $\alpha_l - \alpha_u$ is used in place of $\alpha_u - \alpha_l$ in (5 6) If we say that the switching is from α_u to α_l at $t_1,t_3,t_5,$ and from α_l to α_u at $t_2,t_4,$ we can extend (5.6) to

$$dZ - (\boldsymbol{\lambda}_{Z\Omega}^T\delta\mathbf{x})_{t=t_0} = \sum_{i=1}^{r} (-1)^{i-1}\left(\boldsymbol{\lambda}_{Z\Omega}^T\frac{\partial \mathbf{f}}{\partial \alpha}\right)_{t=ti} \times (\alpha_u - \alpha_l)dt_i \quad (5\ 7)$$

$$= \sum_{i=1}^{r}\frac{\partial Z}{\partial t_i}dt_i$$

Consider now that we have performance index ϕ and p terminal constraints $\boldsymbol{\psi}$, as in Sec 4 We obtain expressions for $d\phi$ and $d\boldsymbol{\psi}$ by appropriate substitution in (5 7) In effect we have transformed the control variable $\alpha(t)$ into a sequence of control parameters t_i We observe that there is also a sequence of inequality constraints on these parameters, namely

$$t_{i+1} \geq t_i \qquad (5\ 8)$$

Problems of this type are considered in Ref 3 In control parameter problems the number of parameters must be at least as great as the number of equality ($\boldsymbol{\psi} = 0$) constraints In this case the number of switching times r must at least equal the number of terminal constraints p One additional switching time is also necessary to extremalize ϕ Thus

$$r \geq p + 1 \qquad (5\ 9)$$

If $r = p + 1$, the optimal $p + 1$ switching times can be found by steepest-ascent It might be possible to find a better stationary solution by using $p + s$, $s > 1$, switching times Using more than $p + 1$ switching times as a nominal should lead to the optimal $p + 1$ switching times if they give a better

** On an extremal solution the combination $(\partial\mathbf{f}/\partial\alpha)^T\boldsymbol{\Lambda} = 0$ off the constraint boundary, where $\boldsymbol{\Lambda} \equiv \boldsymbol{\lambda}_{\phi\Omega} - \boldsymbol{\lambda}_{\psi\Omega}\mathbf{I}_{\psi\psi}^{-1}\mathbf{I}_{\psi\phi}$ Along the boundary from t_1 to t_2, $(\partial\mathbf{f}/\partial\alpha)^T\boldsymbol{\Lambda}$ has the same (opposite) sign as $\partial C/\partial\alpha$ if ϕ is being maximized (minimized) The proper time to come off the boundary is when $[(\partial\mathbf{f}/\partial\alpha)^T\boldsymbol{\Lambda}]_{t=t_2-} = 0$ At this point $\alpha(t_2-)$ is determined by $[S^{(q)}(\mathbf{x},\alpha,t)]_{t=t_2-} = 0$ Since $\boldsymbol{\Lambda}$ is continuous and $\partial\mathbf{f}/\partial\alpha$ is assumed to be a continuous function of α, $\alpha(t_2+) = \alpha(t_2-)$ if $[(\partial\mathbf{f}/\partial\alpha)^T\boldsymbol{\Lambda}]_{t=t_2+}$ also is to be zero, except under rather special circumstances, which are discussed in Chap 4 of Ref 3

†† If the optimal solution possesses a singular arc, the steepest-ascent technique should move toward it H J Kelley has indicated, however, that acceptable convergence was not obtained in his experience Modifications to improve convergence in singular arc problems are being investigated by the authors and others

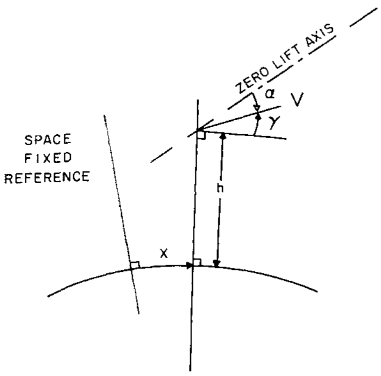

Fig 1 Nomenclature for lifting re-entry vehicle trajectory analysis

result than any distinct $p + s$ times Steepest-ascent can only optimize the parameters assumed to be in the problem Thus, two switching times might become equal in the limit, thereby reducing the number of times from the nominal number, but no additional times can be added by the technique

If the global optimum solution with p components of ψ had $r < p + 1$ switching times it could be found by dropping some of the ψ components This can occur for certain equations of motion with certain initial conditions In this case satisfying a particular subset of the terminal constraints automatically leads to satisfying all of them, so that some were unnecessary In general, however, it will be necessary to choose $r \geq p + 1$

By choosing the t_i as control parameters we rule out the possibility of finding a singular sub-arc Many problems linear in the control variable possess the possibility of two different extremal solutions Steepest-ascent can only find the assumed form We may point out, however, that if a singular arc form is assumed the "singular arc" might shift entirely to a constraint boundary, giving a bang-bang answer The reverse cannot be done

6 Numerical Example with a Control Variable Inequality Constraint

The problem considered here was to find the angle-of-attack program $\alpha(t)$ for a lifting re-entry vehicle to minimize energy loss in bringing the vehicle to a horizontal flight condition at a given altitude in the atmosphere (in this case taken as 250,000 ft) This was to be done subject to the constraint that the resultant aerodynamic force not exceed five times the sea-level weight of the vehicle Since aerodynamic force is a function of the state variables (velocity and altitude) and the control variable (angle of attack), this is a control variable inequality constraint

For a given initial energy and final altitude, minimizing energy loss is the same as maximizing final velocity From previous studies it was apparent that the vehicle would go below 250,000 ft and then climb back up to 250,000 ft for an optimal path Thus the stopping condition used was the *second* time horizontal flight was achieved

The nomenclature of the problem is given in Fig 1, where

V = velocity
γ = flight-path angle relative to local horizontal
h = altitude
x = surface range
α = vehicle angle of attack

The equations of motion are

$$\dot{V} = -(D/m) - g \sin\gamma \quad (6\ 1)$$

$$\dot{\gamma} = (L/mV) + [(V/R + h) - (g/V)] \cos\gamma \quad (6\ 2)$$

$$\dot{h} = V \sin\gamma \quad (6\ 3)$$

$$\dot{x} = \frac{V}{1 + (h/R)} \cos\gamma \quad (6\ 4)$$

where

g = $g_0(R/R + h)^2$ acceleration due to gravity
R = radius of the earth (20,904,000 ft)
D = $C_D(\alpha)\frac{1}{2}\rho(h)V^2S$, drag force
L = $C_L(\alpha)\frac{1}{2}\rho(h)V^2S$, lift force
ρ = $\rho(h)$, atmospheric density (1956 ARDC standard atmosphere used)
$C_D(\alpha)$ = drag coefficient
$C_L(\alpha)$ = lift coefficient
S = reference area
m = vehicle mass

The control variable is $\alpha(t)$ The ballistic coefficient $C_{D\min}S/m$ was taken as 0 15 ft^2 slug^{-1}, and the (C_LS/m) vs (C_DS/m) polar is given in Fig 2 (These parameters were chosen to approximate a proposed Apollo re-entry vehicle)

The unconstrained maximum-velocity trajectory was determined first For this problem the adjoint equations are

$$\dot{\lambda} - \lambda \frac{2D}{mV} + \lambda_\gamma \left[\frac{L}{mV^2} + \cos\gamma\left(\frac{g}{V^2} + \frac{1}{R+h}\right)\right] + \lambda_x \frac{R\cos\gamma}{R+h} + \lambda_h \sin\gamma = 0 \quad (6\ 5)$$

$$\dot{\lambda}_\gamma - \lambda\ g\cos\gamma + \lambda_\gamma \left(\frac{g}{V} - \frac{V}{R+h}\right)\sin\gamma - \lambda_x \frac{RV\sin\gamma}{R+h} + \lambda_h V\cos\gamma = 0 \quad (6\ 6)$$

$$\dot{\lambda}_h - \lambda \left(\frac{D}{m\rho}\frac{d\rho}{dh} - \frac{2g\sin\gamma}{R+h}\right) + \lambda_\gamma \left[\frac{L}{mV\rho}\frac{d\rho}{dh} + \left(\frac{2g}{V(R+h)} - \frac{V}{(R+h)^2}\right)\cos\gamma\right] = 0 \quad (6\ 7)$$

$$\dot{\lambda}_x = 0 \quad (6\ 8)$$

The impulse response function is given by

$$\lambda^T \frac{\partial f}{\partial \alpha} = -\lambda \frac{D}{mC_D}\frac{\partial C_D}{\partial \alpha} + \lambda_\gamma \frac{L}{mVC_L}\frac{\partial C_L}{\partial \alpha} \quad (6\ 9)$$

The performance index was final velocity,

$$\phi = V \quad (6\ 10)$$

The stopping condition used was

$$\Omega = \gamma = 0 \quad (6\ 11)$$

(the second time γ reaches zero), and the (single) terminal constraint function was

$$\psi = h - 250{,}565 \text{ ft} = 0 \quad (6\ 12)$$

The initial conditions (fixed) were

$$V = 36{,}000 \text{ ft-sec}^{-1}$$
$$\gamma = -7\ 5° \quad (6\ 13)$$
$$h = 400{,}000 \text{ ft}$$

Starting with a nominal which hit $\gamma = 0$ (the second time) with $V = 27{,}938$ ft-sec^{-1} and $h = 239{,}360$ ft, a sequence of twenty-one improving solutions produced a velocity of 27,768 ft-sec^{-1} at the altitude of 250,576 ft The resultant trajectory and $\alpha(t)$ program are shown in Figs 3 and 4 respectively The aerodynamic-force history is shown in Fig 5, hitting a peak of 6 55 times sea-level weight Constraining the force to any lower level, then, will reduce the maximum terminal velocity

In the introduction we stated that the present direct method of including inequality constraints in the steepest-ascent

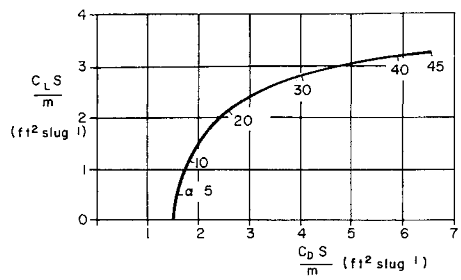

Fig 2 Drag polar of vehicle used in numerical examples

technique has significant advantages over the penalty function approach To demonstrate this the constrained trajectories were calculated using both methods

In the direct method the inequality constraint is

$$F = \frac{(L^2 + D^2)^{1/2}}{mg_0} \leq F_{\max} \tag{6 14}$$

By (3 5) terms are added to the adjoint equations when $F = F_{m\ x}$ The term

$$-\frac{\lambda^T(\partial \mathbf{f}/\partial\alpha)}{\partial F/\partial\alpha}\frac{\partial F}{\partial V} \tag{6 15}$$

must be added to (6 5), and the term

$$-\frac{\lambda^T(\partial \mathbf{f}/\partial\alpha)}{\partial F/\partial\alpha}\frac{\partial F}{\partial h} \tag{6 16}$$

must be added to (6 7), where

$$\begin{aligned}\frac{\partial F}{\partial\alpha} &= \frac{D(\partial C_D/\partial\alpha) + L(\partial C_L/\partial\alpha)}{mg_0(L^2 + D^2)^{1/2}}\\ \frac{\partial F}{\partial V} &= \frac{2F}{V}\\ \frac{\partial F}{\partial h} &= \frac{F}{\rho}\frac{d\rho}{dh}\end{aligned} \tag{6 17}$$

For this problem a particularly simple scheme for handling $\alpha(t)$ at the corners was used It is to be expected that in the $F = F_{\max}$ portion of the trajectory the $\alpha(t)$ program used will be concave upward, thus lying below a linear curve connecting the α values at the corner points The new input $\alpha(t)$ program for each iteration was chosen as linear between the new $\alpha(t_1)$ and the new $\alpha(t_2)$ Although t_1 and t_2 change, in general, on each run, continuity of α is assured As a stationary solution is approached $\delta\alpha(t_1)$ and $\delta\alpha(t_2)$ will tend to zero just as $\delta\alpha(t)$ does elsewhere and the $\alpha(t)$ used when $F = F_{\max}$ will become stationary

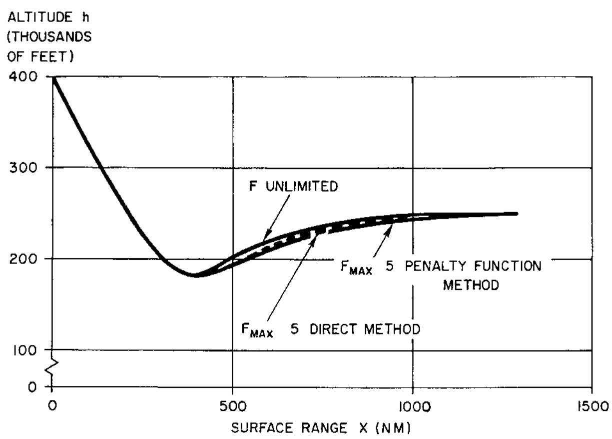

Fig 3 Trajectories for minimum energy loss example

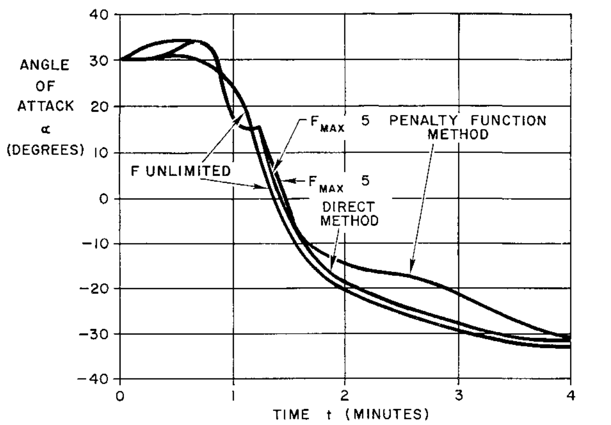

Fig 4 Angle-of-attack programs for minimum energy loss example

For this example $F_m = 5$ was chosen Beginning the successive improvement sequence, a nominal trajectory was found which stopped with $V = 28{,}042$ ft-sec^{-1}, $h = 239{,}731$ ft After a sequence of twenty-one improvements the terminal velocity was 27,704 ft-sec^{-1} and the terminal altitude was 250,558 ft The loss due to the constraint was thus only 64 ft-sec^{-1} The trajectory, shown in Fig 3, is seen to come out of the bounce at a slightly lower altitude as a result of the smaller lift allowed during the bounce The loss in terminal velocity may be attributed to the slightly lower average altitude with a correspondingly slightly higher average drag

The $\alpha(t)$ program and the aerodynamic force history are shown in Figs 4 and 5 respectively The constrained force history is greater than the unconstrained except on the constraint boundary This "makes up" for the constrained period The $\alpha(t)$ program is very similar to the unconstrained program $\alpha(t)$ is smooth at reaching $F = 5$, but has a sharp change after leaving $F = 5$ After coming off the $F = 5$ boundary the α program follows the unconstrained $\alpha(t)$ very closely, slightly above

The nominal $\alpha(t)$ program would have violated the $F \leq 5$ constraint It was automatically modified, on the nominal trajectory, to satisfy the constraint This immediately gave an $\alpha(t)$ program shaped like the constrained one in Fig 4 Every subsequent iteration maintained this shape, with the times of entering onto and leaving the constraint boundary changing only a few thousandths of a minute from run to run

The penalty function technique converts inflight inequality constraints to terminal constraints The usual approach is to introduce a penalty function P, such that

$$\begin{aligned}\dot{P} &= K(F - F_1)^2 \qquad F \geq F_1\\ &= 0 \qquad\qquad\qquad F < F_1\end{aligned} \tag{6 18}$$

where $F \leq F_{\max}$ is the desired inequality constraint K is an arbitrary constant, F_1 is a chosen constant If $F_1 = F_{\max}$ then the inequality constraint will be satisfied only if the final value of P is zero [$P(0) = 0$ is assumed] With steepest-ascent, however, P cannot be controlled if it does not exist, and P is identically zero unless F is greater than F_1 in some region To use $F_1 = F_{\max}$, then, some violation of the constraint must be tolerated F_1 can be chosen as less than $F_{m\ x}$ just as easily, and the resulting maximum value of F can then be made as close to $F_{m\ x}$ as desired

In this example, use of the penalty function P adds the adjoint equation

$$\dot{\lambda}_p = 0 \tag{6 19}$$

Equation (6 5) is modified by the addition of the term

$$2K\lambda_p(F - F_1)(\partial F/\partial V) \tag{6 20}$$

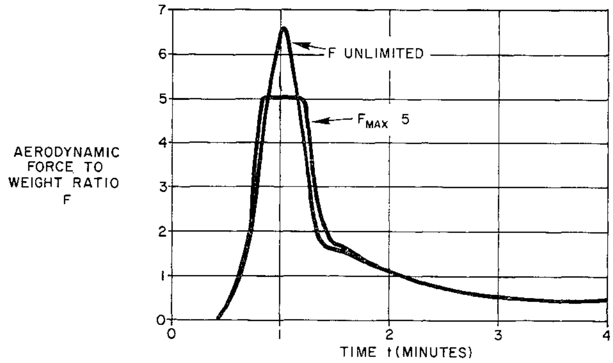

Fig 5 Deceleration histories for minimum energy loss example

when $F > F_1$ The term

$$2K\lambda_p(F - F_1)(\partial F/\partial h) \tag{6 21}$$

must be added to (6 7) when $F > F_1$ and the term

$$2K\lambda_p(F - F_1)\,\frac{L(\partial C_L/\partial \alpha) + D(\partial C_D/\partial \alpha)}{mg_0(L^2 + D^2)^{1/2}} \tag{6 22}$$

must be added to $\boldsymbol{\lambda}^T(\partial \mathbf{f}/\partial \alpha)$ when $F > F_1$ The desired terminal value of P is not known a priori, but must be determined during the improvement sequence as the value which will produce a peak F within the desired tolerance

In contrast to the direct method, $\delta\alpha(t)$ must be determined over the entire trajectory, including the period where $F \cong 5$, when using the penalty function method In fact, the impulse-response functions are usually greatest in the region of largest F, which means the $\delta\alpha(t)$ will be greatest in this region Such indeed is the case in the present problem

Since the improved $\alpha(t)$ program must be generated entirely through steepest-ascent, the desired shape in the $F = 5$ region is not obtained on the nominal, but only after a long sequence of iterations It may be expected that $\alpha(t)$ for $F < 5$ will not converge as rapidly to its optimum value as in the direct method because so much of the "effort" is spent on each improvement in fixing $\alpha(t)$ in the $F \cong 5$ region Some advantage can be gained by a skillful choice of $W(t)$ However, even with studied selections of dP^2 and $W(t)$, eighty-five iterations were needed to produce the penalty function results shown in Figs 3, 4, and 5 On the last improvement the terminal velocity was 27,652 ft-sec^{-1} and the terminal altitude 250,564 ft Thus, 50 ft-sec^{-1} *less* terminal velocity was obtained, and four times as many iterations were required for the penalty function solution compared to the direct method solution We feel this is due chiefly to the absence of a $\delta\alpha(t)$ calculation when on the constraint boundary in the direct method, and that the advantage gained in this way will apply to a very substantial class of problems

7 Numerical Example with a Second-Order State Variable Inequality Constraint

The problem considered here was to find the angle-of-attack program $\alpha(t)$ for a lifting re-entry vehicle to maximize range while not exceeding a prescribed altitude limit after the first dip into the atmosphere The vehicle and initial conditions used were precisely the same as in the previous example The deceleration constraint was omitted, simply because the computer program prepared for the altitude-constrained problem was only intended to verify the state variable constraint theory

In this example the inequality constraint is

$$S = h - h_{\max} \leq 0 \tag{7 1}$$

(after the first time $\gamma = 0$) We observe that

$$\dot{S} = \dot{h} = V \sin\gamma \tag{7 2}$$

does not contain the control variable α We must therefore use

$$\ddot{S} = \dot{V}\sin\gamma + V\dot{\gamma}\cos\gamma = 0 \tag{7 3}$$

as the equation determining α while on the constraint boundary Thus, we have a second-order state variable inequality constraint The adjoint equations when $\ddot{S} = 0$ are given by

$$\frac{d\boldsymbol{\lambda}}{dt} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}}{\partial \alpha}\left(\frac{\partial \ddot{S}}{\partial \alpha}\right)^{-1}\frac{\partial \ddot{S}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = 0 \tag{7 4}$$

An interesting feature of this example is that $\ddot{S} = 0$ is equivalent to $\dot{\gamma} = 0$ when $\dot{S} = 0$ Using $\dot{\gamma} = 0$ to keep $\gamma = 0$ is precisely the same as using $\ddot{S} = 0$ to keep $\dot{S} = 0$ Instead of using $S^{(1)}$ and $S^{(2)}$ themselves as described in Sec 3 we may equally correctly, and more simply, use γ and $\dot{\gamma}$ in their places In fact, by dropping a $1/V$ factor in $\dot{\gamma}$ we may substitute

$$L/m + [(V^2/R + h)] - g = 0 \tag{7 5}$$

for $\dot{S} = \dot{h} = 0$ as the relation which determines α along the constraint boundary This is solved along the boundary to give

$$C_L(\alpha) = -\frac{[(V^2/R + h) - g]}{\rho V^2 S/2m} \tag{7 6}$$

The adjoint equations are modified for the periods on the constraint boundary The term

$$\frac{-\boldsymbol{\lambda}^T(\partial \mathbf{f}/\partial \alpha)}{(L/mC_L)(\partial C_L/\partial \alpha)}\left[\frac{2L}{mV} + \frac{2V}{R + h}\right] \tag{7 7}$$

must be added to (6 5) The term

$$\frac{-\boldsymbol{\lambda}^T(\partial \mathbf{f}/\partial \alpha)}{(L/mC_L)(\partial C_L/\partial \alpha)}\left[\frac{L}{m\rho}\frac{d\rho}{dh} - \frac{V^2}{(R + h)^2} + \frac{2g}{R + h}\right] \tag{7 8}$$

must be added to (6 7)

Only the influence function λ_γ will be discontinuous at the entry corner t_1 The jump is given by

$$\lambda_\gamma(t_1 -) = \lambda_\gamma(t_1 +) - \left[\frac{\lambda\,(t_1+)\left(\dfrac{-D(t_1-) + D(t_1+)}{m}\right) + \lambda_\gamma(t_1+)\left(\dfrac{L(t_1-) - L(t_1+)}{mV}\right)}{\dfrac{L}{m}\left(\dfrac{V^2}{R + h} - g\right)}\,V\right]_{t = t_1 -} \tag{7 9}$$

In this example the performance index was

$$\phi = x \tag{7 10}$$

The stopping condition was

$$\Omega = h - 70{,}000 \text{ ft} = 0 \tag{7 11}$$

There was one additional "terminal" constraint function

$$\psi = h(t_1) - h_{\max} = 0 \tag{7 12}$$

which was handled as described in Sec 4

As in Sec 6, the range maximization was performed both by the direct methods of Sec 4 and by use of an integral penalty function In the latter case we denoted the penalty function as A, which satisfies the differential equation

$$\begin{aligned} \dot{A} &= K(h - h_1)^2 \qquad & h > h_1,\ x > x_1 \\ &= 0 & h < h_1 \text{or } x < x_1 \end{aligned} \tag{7 13}$$

where h_1 was chosen 5000 ft less than $h_{\max}$, and x_1 was taken as the value of x at the bottom of the pull-up The additional adjoint equation is

$$\dot{\lambda}_A = 0 \tag{7 14}$$

The term

$$2K\lambda_A(h - h_1) \tag{7 15}$$

must be added to (6 7) when $h > h_1$, $x > x_1$ The impulse response function is unchanged from (6 9) The terminal value of A is not known a priori; it must be determined in the improvement sequence as the value which will produce a peak h within the desired tolerance Three values of h_{max} were chosen for this example: 250,000 ft, 225,000 ft, and 200,000 ft For both the direct method and the penalty function method a number of trial trajectories were calculated to find one with $h(t_1)$ close to 250,000 ft Since the trajectory of a vehicle entering at parabolic speed is extremely sensitive to small changes in $\alpha(t)$, this task is not trivial The nominals used for h_{max} = 250,000 ft in the two methods were obtained independently of each other

The nominal used in the penalty function approach reached an altitude of 256,000 ft and a surface range of 4863 naut miles After eight iterations the peak altitude was 250,000 ft, the range 4742 naut miles At this point the range increase per iteration was running about 20 naut miles, with the general character of the trajectory not changing

The nominal used for the direct method approach reached 240,000 ft and a surface range of 4661 naut miles After three iterations the peak altitude was 249,300 ft, the range 5379 naut miles Three additional iterations produced a peak altitude of 250,100 ft and a range of 5642 naut miles At this point the range increase per iteration was running about 30 naut miles

For both methods the nominal used for the $h_{m\ x}$ = 225,000 ft case was the last trajectory in the h_{max} = 250,000 ft series This should allow a meaningful comparison of the relative convergence capabilities

In the penalty function approach, seven iterations were used in reducing the peak altitude to 225,000 ft The range by then had shortened to 3401 naut miles After nine more iterations the peak altitude was 224,500 ft, the range 3696 naut miles At this point the rate of improvement was about 10 naut miles per iteration

In the direct approach four iterations brought the peak altitude to 225,100 ft, with the range 4463 naut miles Four subsequent iterations produced no net improvement, indicating that the rate of improvement then was less than 10 naut miles per iteration An excellent approximation to the maximum range trajectory was obtained in just the number of iterations needed to produce the different $h(t_1)$ For h_{max} = 200,000 ft the nominals were the final $h_{m\ x}$ = 225,000 ft trajectories The penalty function used eight iterations to reach 2946 naut mile range, peak altitude 199,700 ft The rate of improvement again was roughly 10 naut miles per iteration The direct method used six iterations to reach 3043 naut mile range, peak altitude 200,000 ft The rate of improvement at this point was only 4 or 5 naut miles per iteration for fixed $h(t_1)$ The difference in result for this h_{max} was negligible, with the direct method reaching a very small gradient, lending a high confidence in the closeness of the approximate optimum to the exact stationary solution

The results just cited are considered fairly conclusive for the problem of this example The direct method is seen to converge substantially faster, as best illustrated in the h_{max} = 250,000 ft case In fact, for the larger values of h_{max}, the maximum range obtained by the penalty function method with a slightly greater amount of computation fell more than 15% less than the direct method results As in the previous example this may be attributed largely to the difference in calculating $\delta\alpha(t)$ along the constraint boundary The angle-of-attack programs from both methods are plotted in Figs 6, 7, and 8 for h_{max} equal to 250, 225, and 200 thousand ft, respectively In addition, the maximum range trajectories from the direct method solutions are plotted together in Fig 9 The h_{max} = 250,000 ft trajectories resulting from the two methods are plotted in Fig 10 Examination of Fig 10 suggests that a big part of the direct method gain is due to the faster pull-up, the minimum altitude in pull-up being 8000 ft higher than in the penalty function solution The direct method trajectory then leaves the h_{max} boundary 400 naut miles further down-range and leaves more smoothly, with an ultimate range differential of 900 naut miles It is likely that a sufficient number of iterations in the penalty function approach would produce approximately the direct method results The advantage of the direct method is that it requires a much smaller number of iterations

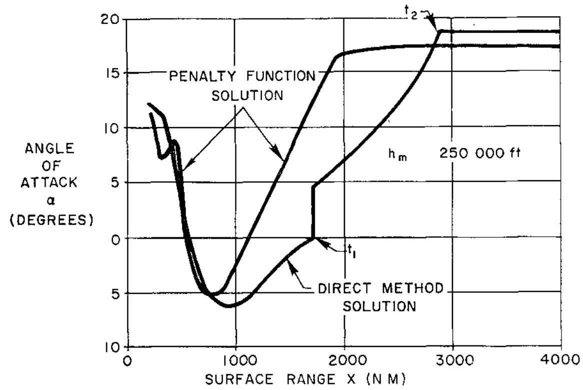

Fig 6 Maximum range example; angle-of-attack programs for $h_{m\ x}$ = 250,000 ft

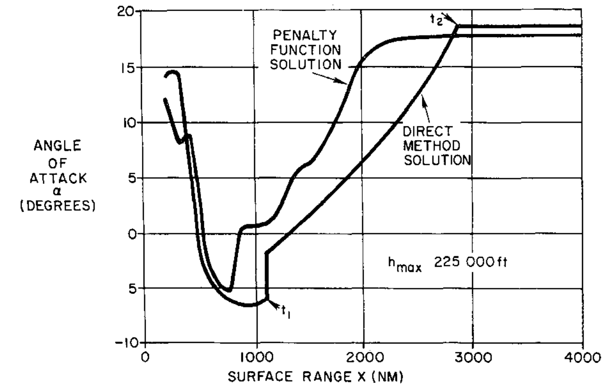

Fig 7 Maximum range example; angle-of-attack programs for $h_{m\ x}$= 225,000 ft

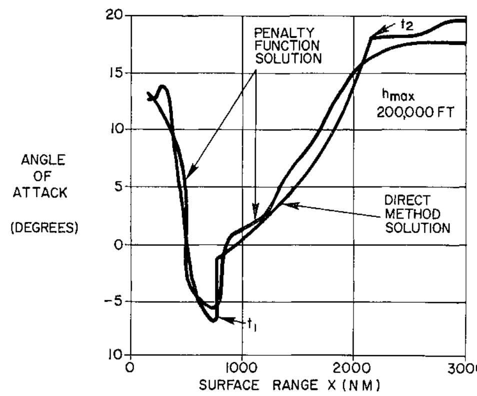

Fig 8 Maximum range example; angle-of-attack programs for h_{max} = 200,000 ft

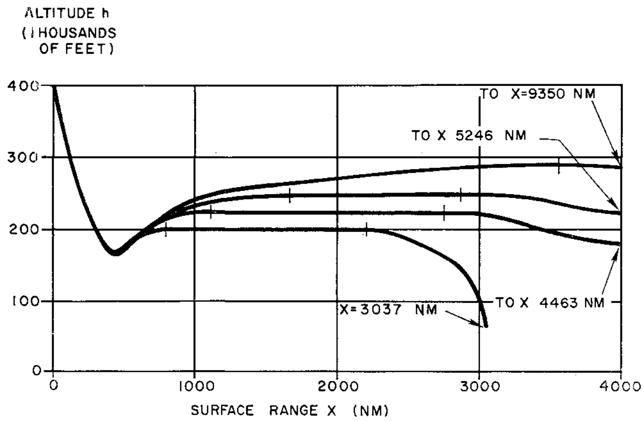

Fig 9 Maximum range example; trajectories from direct method calculations

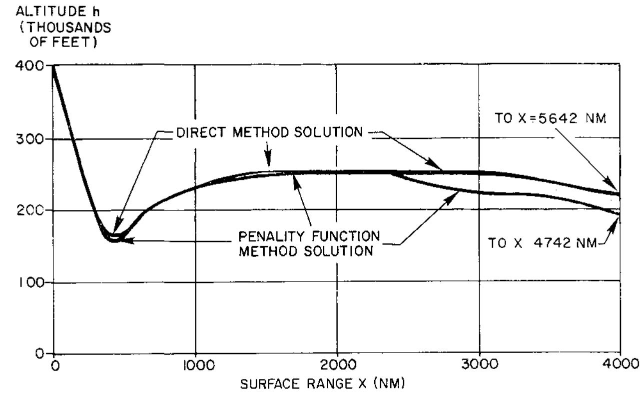

Fig 10 Maximum range example; trajectories from both methods for h_{max} = 250,000 ft

In Figs 5–10 the abscissa is taken as the surface range x This is preferable to time t because of the tremendous variation in velocity V over the flight Because γ is small throughout and the drag differences are small, $x(t)$ will differ very little among the trajectories calculated during the pull-up phase

The α programs resulting from the direct and penalty function approaches are similar in gross structure In neither approach is an exact extremal obtained In particular, we would expect an exact extremal to have a smooth $\alpha(t)$ program However, since the flight path involves an integration of α, variations from the exact extremal $\alpha(t)$ program may produce an almost insignificant change in the integrated result The calculated $\alpha(t)$ programs over the first 200 miles of the trajectories have been purposely omitted because the extremely low dynamic pressure above 275,000 ft alt makes α almost inconsequential there

Although the finer details of the $\alpha(t)$ programs are not generally important, it should be noted that the direct method gives precisely the right form of α for $t_1 < t < t_2$ The only change on an exact extremal would be very small shifts in t_1 and t_2 with a slight displacement of the smooth $\alpha(t)$ arc in the interval The penalty function method has the problem of shaping $\alpha(t)$ for $t_1 \leq t \leq t_2$ as well as for other t, and a substantial portion of the $(dP)^2$ integral is due to $\delta\alpha(t)$ for h in the vicinity of h_{max} This accounts for the relatively slow convergence of the penalty function approach

The h_{max} = 200,000 case allowed the closest comparison of the two $\alpha(t)$ programs Here the penalty function range was 96% of the direct method range It is not difficult to surmise that a major part of the 4% loss was due to not obtaining a smooth $\alpha(t)$ in the $t_1 \leq t \leq t_2$ interval

The most notable difference in the forms of the $\alpha(t)$ programs obtained by the two methods is the discontinuity in $\alpha(t_1)$ in the direct method results It is not clear if the discontinuity would exist on the exact extremal solution In the problem of this section it is clear that $\alpha(t_1+)$ will be non optimal only to the extent that the state $\mathbf{x}(t_1)$ is nonoptimal The question is: Does the exact extremal solution differ from the approximate in just the right way for $\alpha(t_1-)$ to become equal to $\alpha(t_1+)$? The two analytical examples in Ref 4 both have continuous control variables at the entry corner This behavior may hold on exact extremals for a variety of nonlinear problems as well However, even if it does, we may not expect the steepest-ascent approximate optimization to obtain the continuous $\alpha(t_1)$ result because of the particularly delicate relationship required by $S^{(q)}[\mathbf{x},\alpha,t]_{t_1-}$ going to zero just when $S^{(q-1)}[\mathbf{x},t]_{t_1-}$ does The influence functions will generally be discontinuous at t_1 even if the control variable $\alpha(t_1)$ is continuous The example problems in Ref 4 exhibit this feature

One further comparison in the calculation of control program changes is in the use of the weighting function $W(t)$ For the penalty function approach $W(t)$ is useful and can improve convergence Since $\delta\alpha(t)$ must be calculated for all t, however, $W(t)$ should be taken as a smooth function, and cannot be used to sharply distinguish $t < t_1$ from $t > t_2$ as the direct method can The direct method, in fact, was used with W equal to one constant before t_1 and another after t_2 This enabled greater $\delta\alpha(t)$, $t > t_2$ to be made if useful Since nominal solutions in both cases chose α near the maximum L/D value for $t > t_2$, the latter portions of the nominal $\alpha(t)$ programs were hardly changed Since t_2 may be moved substantially on each iteration, it is particularly easy in the direct method to make $\alpha(t_2)$ optimal

A certain analogy may be seen to exist between this problem and the analytic example in Sec 6 of Ref 4 The range could be maximized without specifying a skip altitude limit The resultant trajectory would have a maximum altitude, on the order of several million feet As the altitude limit was decreased the trajectory would still only touch the altitude limit until the altitude analogous to $l/6$ in Ref 4 was reached Any lower value of the limit would result in a finite time spent at the constraint level For this example this critical altitude was experimentally determined to be 291,000 ft The altitude history for this case was added to Fig 9

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